THE LAPLACE TRANSFORM

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LAPLACE TRANSFORM

The Laplace Transform is one of the mathematical tools used for the solution of linear ordinary differential equations.

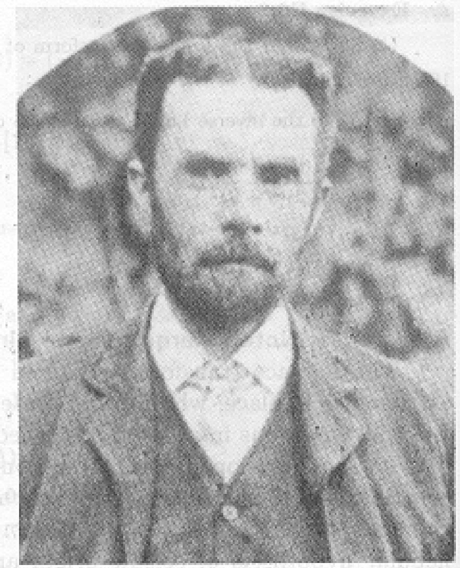
Two attractive features to use Laplace Transform instead of using classical method are:

- ✓ The homogeneous equation and the particular integral of the solution are obtained in one operation.
- ✓ The Laplace Transform converts the differential equation into an algebraic equation in "s". It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the s-domain. The final solution is obtained by taking the inverse Laplace Transform.

A HISTORICAL NOTE

Marquis Pierre-Simon De Laplace
(1749-1827)

- ✓ French mathematician and astronomer;
- ✓ Paper published in 1779 - Applications to differential equations;
- ✓ Traite de Mechanique Celeste.



Oliver Heaviside
(1850-1925)

- ✓ Eccentric British Engineer.
- ✓ Made transatlantic communication possible;
- ✓ The innovator of lowpass filters;
- ✓ Operational mathematics used do solve linear integro-differential equations.

DEFINITION OF LAPLACE TRANSFORM

if $\int_0^{\infty} \left| f(t)e^{-\sigma t} \right| dt < \infty$ for some finite real σ_o then

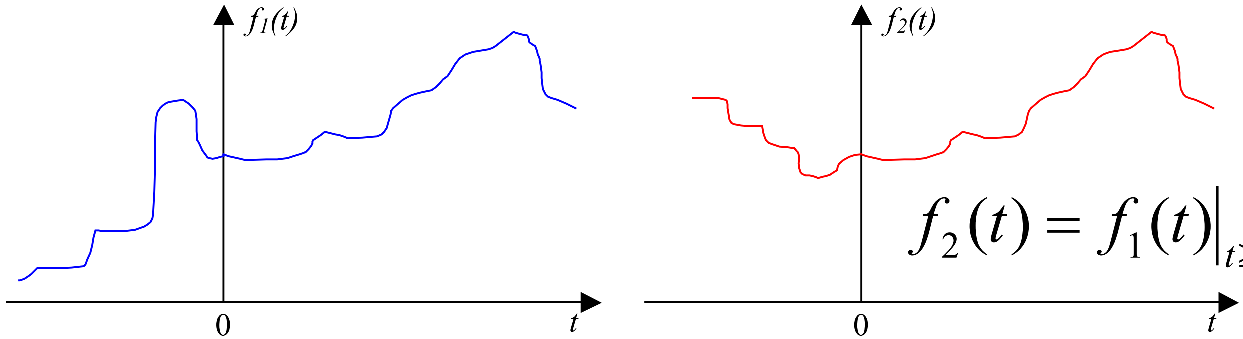
$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \quad \therefore \quad f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds \text{ for } c \geq \sigma_o$$

$$F(s) = L[f(t)] \quad \therefore \quad f(t) = L^{-1}[F(s)]$$

$$f(t) \xrightarrow{L} F(s) \quad \therefore \quad F(s) \xrightarrow{L^{-1}} f(t)$$

- ✓ The variable $s = \sigma + j\omega$ is referred to as the **LAPLACE OPERATOR**.
- ✓ The defining ***F(s)*** as shown in the above equations is also known as the **UNILATERAL LAPLACE TRANSFORM**.

The Unilateral LAPLACE TRANSFORM



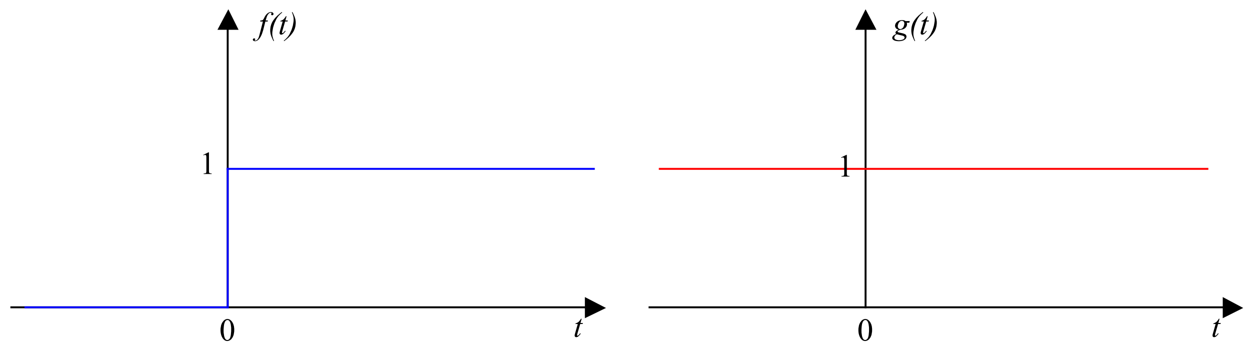
$$f_2(t) = f_1(t)\big|_{t\geq 0}$$

$$f_1(t) \xrightarrow{L} F_1(s) \qquad f_2(t) \xrightarrow{L} F_2(s)$$

$$F_1(s) = F_2(s)$$

- ✓ All information contained in *f(t)* prior to *t=0* is ignored or considered to be zero. This assumption does not impose any limitation on the applications of the Laplace Transform to linear system problems (Physical System → Causal System).

Example - Unit Step



$$f(t) = \mu(t)\begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad \therefore \quad F(s) = G(s)$$

$$F(s) = L[\mu(t)] = \int_0^{+\infty} \mu(t) e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^{+\infty} = \frac{1}{s}$$

$F(s)$ is valid only if $\int_0^{+\infty} \left|\mu(t) e^{-\sigma t}\right| dt = \int_0^{+\infty} \left|e^{-\sigma t}\right| dt < \infty$

which means that $Re\{s\}$ must be greater than zero.

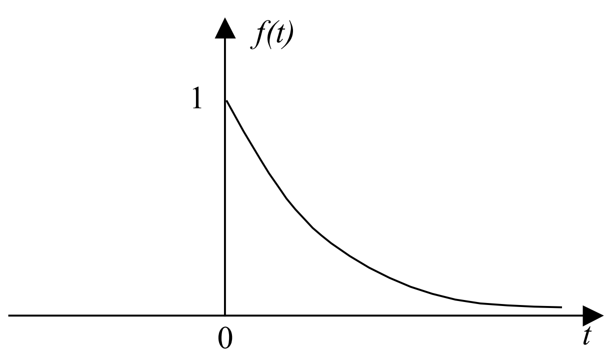
Example - Exponential Pulse

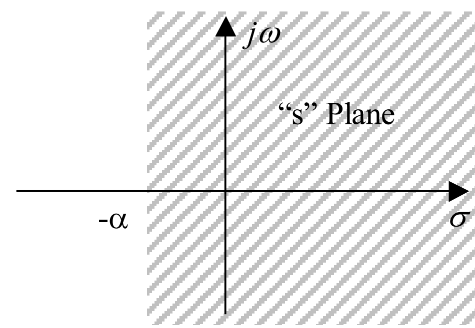
$f(t) = e^{-\alpha t}\mu(t)$; where α is a real constante.

$$F(s) = \int_0^{+\infty} e^{-\alpha t} e^{-st} dt = -\frac{e^{-(s+\alpha)t}}{s+\alpha}\Bigg|_0^{+\infty} = \frac{1}{s+\alpha}$$

$F(s)$ is valid only if $\int_0^{+\infty} \left|e^{-\alpha t} e^{-\sigma t}\right| dt = \int_0^{+\infty} \left|e^{-(\sigma+\alpha)t}\right| dt < \infty$

which means that $Re\{s\}$ must be greater than $-\alpha$.



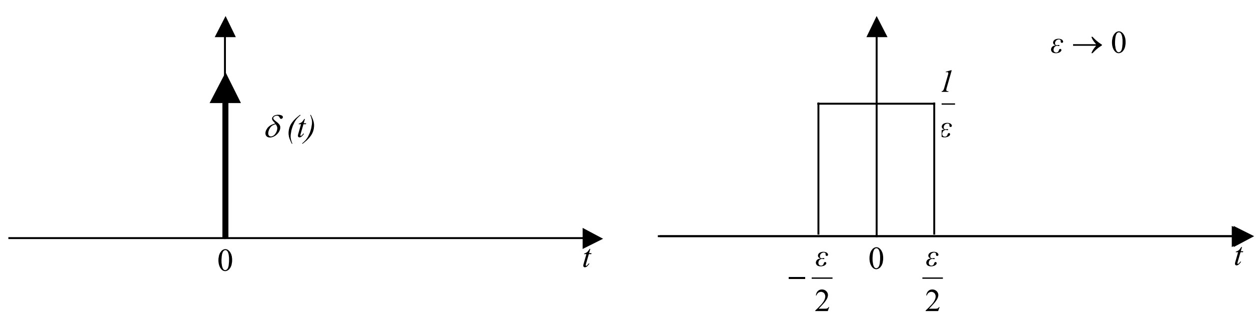


Region of convergence

Example - Unit Impulse

$$f(t) = \delta(t) = \frac{d\mu(t)}{dt} = \begin{cases} 0 & t \neq 0 \\ \int_{-\infty}^{+\infty} \delta(t)dt = 1 & t = 0 \end{cases}$$

$$F(s) = 1 \ \text{ for all } \ s \ \therefore \text{Region of Convergence} = \text{all } \ s$$



A unit impulse and its approximation.

Example - Sinusoid Signal

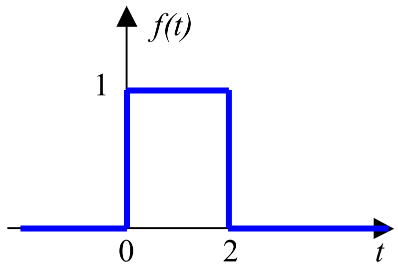
$$f(t) = cos(\omega\, t)\mu(t) = \frac{1}{2}\left[e^{j\omega\, t} + e^{-j\omega\, t}\right]\mu(t)$$

$$F(s) = \frac{1}{2} L\left[e^{j\omega\, t}\mu(t) + e^{-j\omega\, t}\mu(t)\right]$$

$$F(s) = \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] \text{for } Re\{s \pm j\omega\} = Re\{s\} > 0$$

$$F(s) = \frac{s}{s^2 + \omega^2} \therefore \text{Region of Convergence} = Re\{s\} > 0$$

Example – Pulse Signal

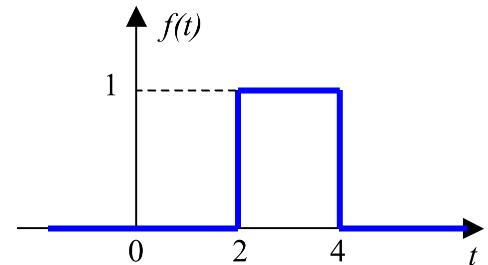


$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$= \int_{0^-}^{2} 1e^{-st}dt + \overbrace{\int_{2}^{\infty} 0e^{-st}dt}^{0}$$

$$= -\left.\frac{e^{-st}}{s}\right|_{0^-}^{2} = \frac{1}{s}(1-e^{-2s})$$

$$F(s) = \frac{1}{s}(1-e^{-2s}) \quad \text{for all} \quad s$$



$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$= \overbrace{\int_{0^-}^{2} 0e^{-st}dt}^{0} + \int_{2}^{4} 1e^{-st}dt + \overbrace{\int_{4}^{\infty} 0e^{-st}dt}^{0}$$

$$= -\left.\frac{e^{-st}}{s}\right|_{2}^{4} = -\frac{1}{s}(e^{-4s} - e^{-2s})$$

$$F(s) = \frac{1}{s}(1-e^{-2s})e^{-2s} \quad \text{for all} \quad s$$

A Short Table of Laplace Transform 1

$$\delta(t) \Leftrightarrow 1$$

$$\mu(t) \Leftrightarrow \frac{1}{s}$$

$$t\,\mu(t) \Leftrightarrow \frac{1}{s^2}$$

$$t^n\,\mu(t) \Leftrightarrow \frac{n!}{s^{n+1}}$$

$$e^{-\alpha t}\mu(t) \Leftrightarrow \frac{1}{s+\alpha}$$

$$t\,e^{-\alpha t}\mu(t) \Leftrightarrow \frac{1}{(s+\alpha)^2}$$

$$t^n\,e^{-\alpha t}\mu(t) \Leftrightarrow \frac{n!}{(s+\alpha)^{n+1}}$$

$$\cos(\omega\,t)\,\mu(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

A Short Table of Laplace Transform 2

$$\sin(\omega t)\,\mu(t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$e^{-\alpha t}\cos(\omega t)\,\mu(t) \Leftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

$$e^{-\alpha t}\sin(\omega t)\,\mu(t) \Leftrightarrow \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$re^{-\alpha t}\cos(\omega t+\theta)\,\mu(t) \Leftrightarrow \frac{(r\cos\theta)s+(r\alpha\cos\theta-r\omega\sin\theta)}{s^2+2\alpha s+(\alpha^2+\omega^2)}$$

$$re^{-\alpha t}\cos(\omega t+\theta)\,\mu(t) \Leftrightarrow \frac{0.5\,r\,e^{j\theta}}{s+\alpha-j\omega}+\frac{0.5\,r\,e^{-j\theta}}{s+\alpha+j\omega}$$

$$re^{-\alpha t}\cos(\omega t+\theta)\,\mu(t) \Leftrightarrow \frac{As+B}{s^2+2\alpha s+C}$$

$$r=\sqrt{\frac{A^2C+B^2-2AB\alpha}{C-\alpha^2}}\;;\theta=\tan^{-1}\frac{\alpha A-B}{A\sqrt{C-\alpha^2}} \qquad \omega=\sqrt{C-\alpha^2}$$

$$e^{-\alpha t}\left[A\cos(\omega t)+\frac{B-\alpha A}{\omega}\sin(\omega t)\right]\mu(t) \Leftrightarrow \frac{As+B}{s^2+2\alpha s+C}$$

$$\omega=\sqrt{C-\alpha^2}$$

The Laplace Transform Properties 1

Property	$f(t)$	$F(s)$
Addition	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
Scalar Multiplication	$k\ f(t)$	$k\ F(s)$
Time Differentiation	$\dfrac{d\ f(t)}{dt}$	$sF(s) - f(0^-)$
	$\dfrac{d^2\ f(t)}{dt^2}$	$s^2F(s) - s\ f(0^-) - \overset{*}{f}(0^-)$
	$\dfrac{d^3\ f(t)}{dt^3}$	$s^3F(s) - s^2 f(0^-) - s\ \overset{*}{f}(0^-) - \overset{**}{f}(0^-)$
Time Integration	$\int_{0^-}^{t} f(\tau)d\tau$	$\dfrac{1}{s}F(s)$
	$\int_{-\infty}^{t} f(\tau)d\tau$	$\dfrac{1}{s}F(s) + \dfrac{1}{s}\int_{-\infty}^{0^-} f(t)dt$
Time Shift	$f(t-t_o)\mu(t-t_o)$	$F(s)e^{-s t_o} \quad t_o \geq 0$
Frequency Shift	$f(t)e^{\pm \alpha t}$	$F(s \mp \alpha)$

The Laplace Transform Properties 2

Frequency Differentiation	$-t\ f(t)$	$\dfrac{d\ F(s)}{ds}$
Frequency Integration	$\dfrac{f(t)}{t}$	$\int_s^{\infty} F(z)dz$
Scaling	$f(\alpha\, t) \therefore \alpha \geq 0$	$\dfrac{1}{\alpha} F\left(\dfrac{s}{\alpha}\right)$
Time Convolution	$f_1(t) * f_2(t)$	$F_1(s)\, F_2(s)$
Frequency Convolution	$f_1(t)\ f_2(t)$	$\dfrac{1}{2\,\pi\, j} F_1(s) * F_2(s)$
Initial Value	$f(0^+)$	$lim_{s \to \infty} sF(s) \therefore (n > m)$
Final Value	$f(\infty)$	$lim_{s \to 0} s\ F(s)$
		Poles of $s\ F(s)$ in LHP

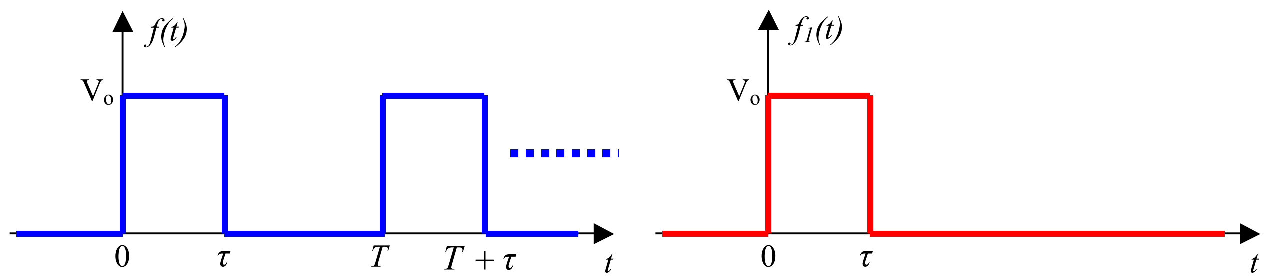
Periodic Function

$$f(t) = f(t - nT) \qquad n = 0, 1, 2, \ldots$$

$$f_1(t) = [\mu(t) - \mu(t - T)]f(t)$$

$$F(s) = \frac{1}{1 - e^{-Ts}} F_1(s) \qquad F_1(s) = L[f_1(t)]$$

Example:

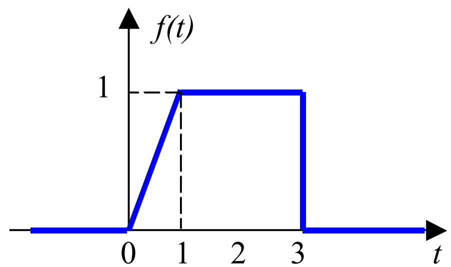


$$f(t) = \sum_{n=0}^{\infty} f_1(t - nT)$$

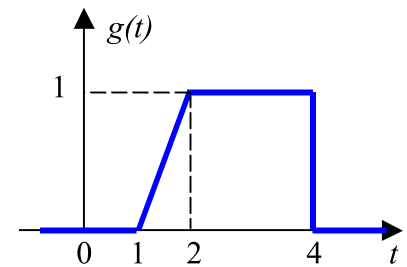
$$f_1(t) = V_o[\mu(t) - \mu(t - \tau)] \xrightarrow{L} F_1(s) = \frac{V_o}{s}(1 - e^{-s\tau})$$

$$F(s) = \frac{V_o}{s} \frac{1 - e^{-s\tau}}{1 - e^{-Ts}}$$

Example – Using unit step function to compose signal and time shifting



$$f(t) = t\,\mu(t) - t\,\mu(t-1)$$
$$+\,\mu(t-1) - \mu(t-3)$$
$$f(t) = t\,\mu(t) - (t-1)\,\mu(t-1)$$
$$-\,\mu(t-3)$$
$$F(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-3s}$$



$$g(t) = (t-1)\,\mu(t-1) - (t-1)\mu(t-2)$$
$$+\,\mu(t-2) + \mu(t-4)$$
$$g(t) = (t-1)\,\mu(t-1) - (t-2)\mu(t-2)$$
$$-\,\mu(t-4)$$
$$G(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

$$g(t) = f(t-1)\,\mu(t-1)$$
$$G(s) = F(s)e^{-s}$$

Example – Time shifting

$$F(s) = \frac{s+3+5e^{-2s}}{(s+1)(s+2)}$$

$$F(s) = \underbrace{\frac{s+3}{(s+1)(s+2)}}_{F_1(s)} + \underbrace{\frac{5e^{-2s}}{(s+1)(s+2)}}_{F_2(s)e^{-2s}}$$

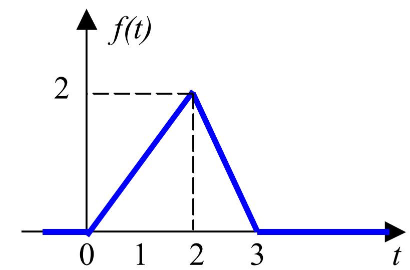
$$F_1(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

$$F_2(s) = \frac{5}{(s+1)(s+2)} = \frac{5}{(s+1)} - \frac{5}{(s+2)}$$

$$f_1(t) = (2e^{-t} - e^{-2t})u(t)$$

$$f_2(t) = 5(e^{-t} - e^{-2t})u(t)$$

$$f(t) = (2e^{-t} - e^{-2t})u(t) + 5[e^{-(t-2)} - e^{-2(t-2)}]u(t-2)$$



$$f(t) = t\,\mu(t) - t\,\mu(t-2) - 2(t-3)\,\mu(t-2) + 2(t-3)\,\mu(t-3)$$

$$f(t) = t\,\mu(t) - 3(t-2)\mu(t-2) + 2(t-3)\,\mu(t-3)$$

$$F(s) = \frac{1}{s^2}\left[1 - 3e^{-2s} + 2e^{-3s}\right]$$

Rational Functions and Partial Fraction Expansion

$$F(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s + b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s + a_0}=\frac{P(s)}{Q(s)}$$

$F(s)$ is improper if $m \geq n$ and proper if $m < n$.

An improper function can always be separated into the sum of a polynomial in *s* and a proper function.

Example:

$$F(s)=\frac{2s^3+9s^2+11s+2}{s^2+4s+3}$$

$$F(s)=\underbrace{2s+1}_{\text{Polynomial in } s}+\underbrace{\frac{s-1}{s^2+4s+3}}_{\text{Proper Function}}$$

$$\begin{array}{r|l} 2s^3+9s^2+11s+2 & s^2+4s+3 \\ \hline \underline{2s^3+8s^2+6s} & 2s+1 \\ s^2+5s+2 & \\ \underline{s^2+4s+3} & \\ s-1 & \end{array}$$

Partial Fraction Expansion
Method of Clearing Fraction

$$F(s)=\frac{s^3+3s^2+4s+6}{(s+1)(s+2)(s+3)^2}=\frac{k_1}{s+1}+\frac{k_2}{s+2}+\frac{k_3}{s+3}+\frac{k_4}{(s+3)^2}$$

To determine the unknowns terms, we clear fractions by multiplying both sides by $(s+1)(s+2)(s+3)^2$ to obtain:

$$s^3+3s^2+4s+6=k_1(s^3+8s^2+21s+18)+k_2(s^3+7s^2+15s+9)$$
$$+k_3(s^3+6s^2+11s+6)+k_4(s^2+3s+2)$$
$$=s^3(k_1+k_2+k_3)+s^2(8k_1+7k_2+6k_3+k_4)$$
$$+s(21k_1+15k_2+11k_3+3k_4)+(18k_1+9k_2+6k_3+2k_4)$$

$$\begin{aligned} k_1+k_2+k_3 &= 1\\ 8k_1+7k_2+6k_3+k_4 &= 3\\ 21k_1+15k_2+11k_3+3k_4 &= 4\\ 18k_1+9k_2+6k_3+2k_4 &= 6 \end{aligned}$$

$$k_1=1,\ k_2=-2,\ k_3=2,\ k_4=-3$$

$$F(s)=\frac{1}{s+1}-\frac{2}{s+2}+\frac{2}{s+3}-\frac{3}{(s+3)^2}$$

Straightforward and applicable to all situations

Partial Fraction Expansion
The Heaviside "Cover-Up"

Cover-Up - Unrepeated Factor of *Q(s)*

$$F(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{P(s)}{Q(s)}\qquad m<n$$

$$=\frac{P(s)}{(s+\lambda_1)(s+\lambda_2)\ldots(s+\lambda_n)}=\frac{k_1}{s+\lambda_1}+\frac{k_2}{s+\lambda_2}+\ldots+\frac{k_n}{s+\lambda_n}$$

$$k_r=(s+\lambda_r)F(s)\Big|_{s=-\lambda_r}\qquad r=1,2,\ldots,n$$

Example:

$$F(s)=\frac{2s^2+9s-11}{(s+1)(s-2)(s+3)}=\frac{k_1}{(s+1)}+\frac{k_2}{(s-2)}+\frac{k_3}{(s+3)}=\boxed{\frac{3}{(s+1)}+\frac{1}{(s-2)}-\frac{2}{(s+3)}}$$

$$k_1=\cancel{(s+1)}\frac{2s^2+9s-11}{\cancel{(s+1)}(s-2)(s+3)}\Bigg|_{s=-1}=\frac{2-9-11}{(-1-2)(-1+3)}=\frac{-18}{-6}=3$$

$$k_2=\cancel{(s-2)}\frac{2s^2+9s-11}{(s+1)\cancel{(s-2)}(s+3)}\Bigg|_{s=2}=\frac{8+18-11}{(2+1)(2+3)}=\frac{15}{15}=1$$

$$k_3=\cancel{(s+3)}\frac{2s^2+9s-11}{(s+1)(s-2)\cancel{(s+3)}}\Bigg|_{s=-3}=\frac{18-27-11}{(-3+1)(-3-2)}=\frac{-20}{10}=-2$$

Partial Fraction Expansion
The Heaviside “Cover-Up”

Cover-Up - Complex Factor in *F(s)*

Example:

$$F(s)=\frac{4s^2+2s+18}{(s+1)(s^2+4s+13)}=\frac{4s^2+2s+18}{(s+1)(s+2-j3)(s+2+j3)}$$

$$F(s)=\frac{k_1}{s+1}+\frac{k_2}{s+2-j3}+\frac{k_3}{s+2+j3}$$

$$k_1=\cancel{(s+1)}\left.\frac{4s^2+2s+18}{\cancel{(s+1)}(s^2+4s+13)}\right|_{s=-1}=2$$

$$k_2=\cancel{(s+2-j3)}\left.\frac{4s^2+2s+18}{(s+1)\cancel{(s+2-j3)}(s+2+j3)}\right|_{s=-2+j3}=1+j2=\sqrt{5}e^{j63.43^o}$$

$$k_3=\cancel{(s+2+j3)}\left.\frac{4s^2+2s+18}{(s+1)(s+2-j3)\cancel{(s+2+j3)}}\right|_{s=-2-j3}=1-j2=\sqrt{5}e^{-j63.43^o}$$

$$F(s)=\frac{2}{s+1}+\frac{\sqrt{5}e^{j63.43^o}}{(s+2-j3)}+\frac{\sqrt{5}e^{-j63.43^o}}{(s+2+j3)}$$

Partial Fraction Expansion
The Heaviside "Cover-Up"

Cover-Up - Quadratic Factor

Example:

$$F(s)=\frac{4s^2+2s+18}{(s+1)(s^2+4s+13)}=\frac{k_1}{s+1}+\frac{c_1s+c_2}{s^2+4s+13}$$

$$k_1=\cancel{(s+1)}\left.\frac{4s^2+2s+18}{\cancel{(s+1)}(s^2+4s+13)}\right|_{s=-1}=2$$

$$\frac{4s^2+2s+18}{(s+1)(s^2+4s+13)}=\frac{2}{s+1}+\frac{c_1s+c_2}{s^2+4s+13}$$

$$4s^2+2s+18=2(s^2+4s+13)+(c_1s+c_2)(s+1)$$

$$4s^2+2s+18=(2+c_1)s^2+(8+c_1+c_2)s+(26+c_2)$$

$$c_1=2\,,\quad c_2=-8$$

$$\boxed{F(s)=\frac{2}{s+1}+\frac{2s-8}{s^2+4s+13}}$$

Partial Fraction Expansion
The Heaviside "Cover-Up"

Cover-Up - Quadratic Factor - Short-Cuts

Example 1:

$$F(s)=\frac{4s^2+2s+18}{(s+1)(s^2+4s+13)}=\frac{k_1}{s+1}+\frac{c_1 s+c_2}{s^2+4s+13}$$

$$k_1=\cancel{(s+1)}\left.\frac{4s^2+2s+18}{\cancel{(s+1)}(s^2+4s+13)}\right|_{s=-1}=2$$

$$\left.\frac{4s^2+2s+18}{(s+1)(s^2+4s+13)}\right|_{s=0}=\left[\frac{2}{s+1}+\frac{c_1 s+c_2}{s^2+4s+13}\right]_{s=0}$$

$$\frac{18}{13}=2+\frac{c_2}{13}\quad\therefore\quad c_2=-8$$

$$\left.\left\{s\left[\frac{4s^2+2s+18}{(s+1)(s^2+4s+13)}\right]\right\}\right|_{s\to\infty}=\left.\left\{s\left[\frac{2}{s+1}+\frac{c_1 s+c_2}{s^2+4s+13}\right]\right\}\right|_{s\to\infty}$$

$$4=2+c_1\quad\therefore\quad c_1=2\quad\therefore\quad \boxed{F(s)=\frac{2}{s+1}+\frac{2s-8}{s^2+4s+13}}$$

Partial Fraction Expansion
The Heaviside "Cover-Up"

Cover-Up - Quadratic Factor - Short-Cuts

Example 2: $$F(s)=\frac{2s^2+4s+5}{s(s^2+2s+5)}=\frac{k_1}{s}+\frac{c_1s+c_2}{s^2+2s+5}$$

$$k_1=\cancel{s}\frac{2s^2+4s+5}{\cancel{s}(s^2+2s+5)}\Bigg|_{s=0}=1 \quad \therefore \quad \cancel{F(0)=\infty}$$

$$\frac{2s^2+4s+5}{s(s^2+2s+5)}\Bigg|_{s=1}=\left[\frac{1}{s}+\frac{c_1s+c_2}{s^2+2s+5}\right]_{s=1}$$

$$\frac{11}{8}=1+\frac{c_1+c_2}{8} \quad \therefore \quad c_1+c_2=3$$

$$\left\{s\left[\frac{2s^2+4s+5}{s(s^2+2s+5)}\right]\right\}\Bigg|_{s\to\infty}=\left\{s\left[\frac{1}{s}+\frac{c_1s+c_2}{s^2+2s+5}\right]\right\}\Bigg|_{s\to\infty}$$

$$2=1+c_1 \quad \therefore \quad c_1=1 \quad \therefore \quad c_2=2 \quad \rightarrow \quad \boxed{F(s)=\frac{1}{s}+\frac{s+2}{s^2+2s+5}}$$

Partial Fraction Expansion

Repeated Factor in *Q(s)*

$$F(s) = \frac{P(s)}{(s+\lambda)^r (s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_j)}$$

$$F(s) = \overbrace{\frac{a_0}{(s+\lambda)^r} + \frac{a_1}{(s+\lambda)^{r-1}} + \ldots + \frac{a_{r-1}}{(s+\lambda)}}^{\text{Repeated Factors}}$$

$$+ \underbrace{\frac{k_1}{s+\alpha_1} + \frac{k_2}{s+\alpha_2} + \ldots + \frac{k_j}{s+\alpha_j}}_{\text{Unrepeated Factors} \rightarrow \text{Heaviside}}$$

$$(s+\lambda)^r F(s)\Big|_{s=-\lambda} = a_0$$

$$\frac{d}{ds}\left[(s+\lambda)^r F(s)\right]\Big|_{s=-\lambda} = a_1$$

$$a_i = \frac{1}{i!}\frac{d^i}{ds^i}\left[(s+\lambda)^r F(s)\right]\Big|_{s=-\lambda} \quad , \quad i = 0, 1, \ldots, r-1$$

Partial Fraction Expansion

Repeated Factor in *Q(s)* - Example

$$F(s)=\frac{4s^3+16s^2+23s+13}{(s+1)^3(s+2)}$$

$$F(s)=\frac{2}{(s+1)^3}+\frac{1}{(s+1)^2}+\frac{3}{(s+1)}+\frac{1}{s+2}$$

$$F(s)=\frac{a_0}{(s+1)^3}+\frac{a_1}{(s+1)^2}+\frac{a_2}{(s+1)}+\frac{k}{s+2}$$

$$k=\cancel{(s+2)}\left.\frac{4s^3+16s^2+23s+13}{(s+1)^3\cancel{(s+2)}}\right|_{s=-2}=1$$

$$a_0=\cancel{(s+1)^3}\left.\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\right|_{s=-1}=2$$

$$a_1=\frac{d}{ds}\left.\left[\cancel{(s+1)^3}\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\right]\right|_{s=-1}=1$$

$$a_3=\frac{1}{2!}\frac{d^2}{ds^2}\left.\left[\cancel{(s+1)^3}\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\right]\right|_{s=-1}=3$$

Partial Fraction Expansion

Mixture of the Heaviside and Clearing Fractions

$$F(s)=\frac{4s^3+16s^2+23s+13}{(s+1)^3(s+2)}=\frac{a_0}{(s+1)^3}+\frac{a_1}{(s+1)^2}+\frac{a_2}{(s+1)}+\frac{k}{s+2}$$

$$k=\cancel{(s+2)}\left.\frac{4s^3+16s^2+23s+13}{(s+1)^3\cancel{(s+2)}}\right|_{s=-2}=1$$

$$a_0=\cancel{(s+1)^3}\left.\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\right|_{s=-1}=2$$

Multiply both sides by $(s+1)^3(s+2)$, yields

$$4s^3+16s^2+23s+13=2(s+2)+a_1(s+1)(s+2)+a_2(s+1)^2(s+2)+(s+1)^3$$

$$=(1+a_2)s^3+(a_1+4a_2+3)s^2+(5+3a_1+5a_2)s+(4+2a_1+2a_2+1)$$

$$\left.\begin{aligned}1+a_2&=4\\a_1+4a_2+3&=16\end{aligned}\right\}\ \text{or}\ \left.\begin{aligned}5+3a_1+5a_2&=23\\4+2a_1+2a_2+1&=13\end{aligned}\right\}\Rightarrow\begin{aligned}a_1&=1\\a_2&=3\end{aligned}$$

$$F(s)=\frac{2}{(s+1)^3}+\frac{1}{(s+1)^2}+\frac{3}{(s+1)}+\frac{1}{s+2}$$

Partial Fraction Expansion

Mixture of the Heaviside and Short Cuts

$$F(s)=\frac{4s^3+16s^2+23s+13}{(s+1)^3(s+2)}=\frac{a_0}{(s+1)^3}+\frac{a_1}{(s+1)^2}+\frac{a_2}{(s+1)}+\frac{k}{s+2}$$

$$k=\cancel{(s+2)}\frac{4s^3+16s^2+23s+13}{(s+1)^3\cancel{(s+2)}}\Bigg|_{s=-2}=1$$

$$a_0=\cancel{(s+1)^3}\frac{4s^3+16s^2+23s+13}{\cancel{(s+1)^3}(s+2)}\Bigg|_{s=-1}=2$$

Multiply both sides by s and then let $s\to\infty$, yields

$$4=a_2+1\Rightarrow a_2=3$$

There is now only one unknown a_1, which can be readily found by setting to any convenient value, say $=0$.

$$13/2=2+a_1+3+1/2\Rightarrow a_1=1$$

$$F(s)=\frac{2}{(s+1)^3}+\frac{1}{(s+1)^2}+\frac{3}{(s+1)}+\frac{1}{s+2}$$

Partial Fraction Expansion

Improper *F(s)* with *m=n*

$$F(s)=\frac{b_n s^n + b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

$$F(s)=b_n+\frac{k_1}{s+\lambda_1}+\frac{k_2}{s+\lambda_2}+\ldots+\frac{k_n}{s+\lambda_n}$$

The coefficients $k_1, k_2, \ldots k_n$ are computed as if $F(s)$ were proper.

Example:

$$F(s)=\frac{3s^2+9s-20}{s^2+s-6}=\frac{3s^2+9s-20}{(s-2)(s+3)} \quad \therefore m=n=2 \text{ with } b_n=b_2=3$$

$$F(s)=\frac{3s^2+9s-20}{(s-2)(s+3)}=3+\frac{k_1}{s-2}+\frac{k_2}{s+3}$$

$$k_1=\cancel{(s-2)}\frac{3s^2+9s-20}{\cancel{(s-2)}(s+3)}\bigg|_{s=2}=\frac{12+18-20}{(2+3)}=\frac{10}{5}=2$$

$$k_3=\cancel{(s+3)}\frac{3s^2+9s-20}{(s-2)\cancel{(s+3)}}\bigg|_{s=-3}=\frac{27-27-20}{(-3-2)}=\frac{-20}{-5}=4$$

$$\boxed{F(s)=3+\frac{2}{s-2}+\frac{4}{s+3}}$$

Modified Partial Fractions

Often we require the form $\frac{ks}{(s+\lambda)^r}$ rather than $\frac{k}{(s+\lambda)^r}$. This can be achived by expanding $F(s)/s$ into partial fractions.

Example:

$$F(s)=\frac{5s^2+20s+18}{(s+2)(s+3)^2}$$

$$\frac{F(s)}{s}=\frac{5s^2+20s+18}{s(s+2)(s+3)^2}=\frac{a_1}{s}+\frac{a_2}{s+2}+\frac{a_3}{(s+3)}+\frac{a_4}{(s+3)^2}$$

$$a_1=s\left.\frac{5s^2+20s+18}{s(s+2)(s+3)^2}\right|_{s=0}=\frac{18}{2(3)^2}=\frac{18}{18}=1$$

$$a_2=\cancel{(s+2)}\left.\frac{5s^2+20s+18}{s\cancel{(s+2)}(s+3)^2}\right|_{s=-2}=\frac{20-40+18}{-2(1)}=\frac{-2}{-2}=1$$

$$a_4=\cancel{(s+3)^2}\left.\frac{5s^2+20s+18}{s(s+2)\cancel{(s+3)^2}}\right|_{s=-3}=\frac{45-60+18}{-3(-1)}=\frac{3}{3}=1$$

$$\left\{\cancel{s}\left[\frac{5s^2+20s+18}{\cancel{s}(s+2)(s+3)^2}\right]\right\}_{s\to\infty}=\left\{s\left[\frac{1}{s}+\frac{1}{s+2}+\frac{a_3}{(s+3)}+\frac{1}{(s+3)^2}\right]\right\}_{s\to\infty}$$

$$a_3=-2 \quad \therefore \frac{F(s)}{s}=\frac{1}{s}+\frac{1}{s+2}-\frac{2}{(s+3)}+\frac{1}{(s+3)^2}$$

$$F(s)=1+\frac{s}{s+2}-\frac{2s}{s+3}+\frac{s}{(s+3)^2}$$

Finding the Inverse Transform

Examples:

$$F(s)=\frac{7s-6}{s^2-s-6}=\frac{7s-6}{(s+2)(s-3)}=\frac{4}{s+2}+\frac{3}{s-3}$$

$$f(t)=\left(4e^{-2t}+3e^{3t}\right)u(t)$$

$$F(s)=\frac{2s^2+5}{s^2+3s+2}=\overbrace{\frac{2s^2+5}{(s+1)(s+2)}}^{\text{Improper}}=2+\frac{7}{s+1}-\frac{13}{s+2}$$

$$f(t)=2\,\delta(t)+\left(7e^{-t}-13e^{-2t}\right)u(t)$$

$$F(s)=\frac{6(s+34)}{s(s^2+10s+34)}=\frac{6}{s}+\frac{5e^{j126.9^o}}{s+5-j3}+\frac{5e^{-j126.9^o}}{s+5+j3}=\frac{6}{s}+\frac{-6s-54}{s^2+10s+34}$$

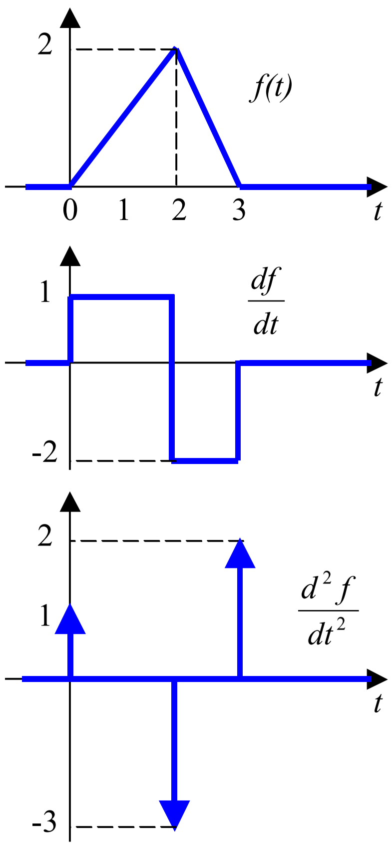
$$f(t)=\left[6+10e^{-5t}\cos(3t+126.9^o)\right]u(t)$$

$$F(s)=\frac{8s+10}{(s+1)(s+3)^3}=\frac{2}{s+1}-\frac{2}{s+2}-\frac{2}{(s+2)^2}+\frac{6}{(s+2)^3}$$

$$f(t)=\left[2e^{-t}-(2+2t-3t^2)e^{-2t}\right]u(t)$$

The Time-Differentiation Property

Example: Find the Laplace Transform of the signal *f(t)* in Figure below using table and the Time-Differentiation and Time-Shifting properties of the Laplace Transform.



$$\frac{d^2 f}{dt^2} = \delta(t) - 3\,\delta(t-2) + 2\,\delta(t-3)$$

$$L\left(\frac{d^2 f}{dt^2}\right) = L\left[\delta(t) - 3\delta\,(t-2) + 2\delta\,(t-3)\right]$$

$$s^2 F(s) - 0 - 0 = 1 - 3e^{-2s} + 2e^{-3s}$$

$$F(s) = \frac{1}{s^2}\left(1 - 3e^{-2s} + 2e^{-3s}\right)$$

Initial and Final Value

The initial value theorem states that *f(t)* and its derivative *df/dt* are both Laplace Transformable, then $f(0^+) = \lim_{s \to \infty} sF(s)$ provided that the limit on the right-hand side of former equation exists. Comment: The Initial value theorem should be applied only if *F(s)* is strictly proper, because for $m \geq n$, $\lim_{s \to \infty} sF(s)$ does not exist, and the theorem does not apply.

The final value theorem states that if both *f(t)* and *df/dt* are Laplace Transformable, then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ provided that *sF(s)* has no poles in the right-half plain (RHP) or on the imaginary axis. Comment: If there is a pole on the imaginary axis, then $\lim_{s \to 0} sF(s)$ does not exist. If there is a pole in the RHP, $\lim_{t \to \infty} f(t)$ does not exist.

Initial and Final Value (Examples)

$$F(s) = \frac{2}{1 - e^{-s}}$$

$$f(0^+) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} = s\frac{2}{1 - e^{-s}} = \infty$$

$$f(\infty) = \lim_{s \to 0} sF(s) = \text{does not exist}$$

$$F(s) = \frac{2s^2}{s^2 + s + 4}$$

$$f(0^+) = \lim_{s \to \infty} sF(s) = \text{does not exist}$$

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} = s\frac{2s^2}{s^2 + s + 4} = 0$$

$$F(s) = \frac{10(2s + 3)}{s(s^2 + 2s + 5)}$$

$$f(0^+) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{10(2s + 3)}{(s^2 + 2s + 5)} = 0$$

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{10(2s + 3)}{(s^2 + 2s + 5)} = 6$$

$$F(s) = \frac{2s^2}{s^2 - s + 4}$$

$$f(0^+) = \lim_{s \to \infty} sF(s) = \text{does not exist}$$

$$f(\infty) = \lim_{s \to \infty} f(t) = \text{does not exist}$$

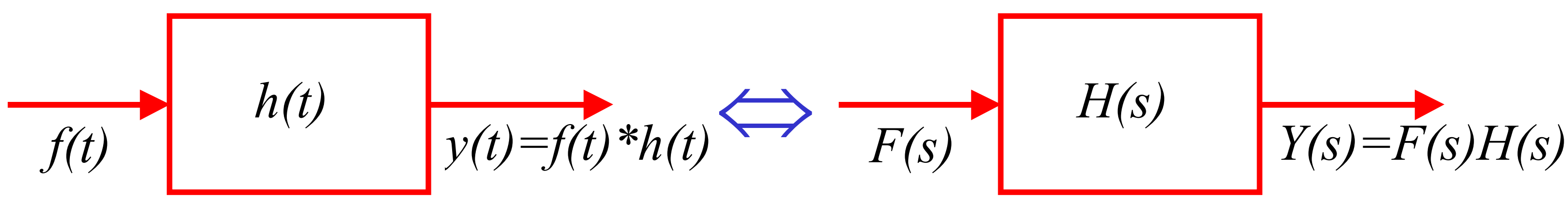
Time Convolution and Frequency Convolution

$$f_1(t) \Leftrightarrow F_1(s) \quad \text{and} \quad f_2(t) \Leftrightarrow F_2(s)$$

$$f_1(t) * f_2(t) \Leftrightarrow F_1(s)F_2(s)$$

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi j}\left[F_1(s) * F_2(s)\right]$$

Time Convolution for Zero-State System- Example



$$f(t) = e^{-2t}u(t) \Leftrightarrow \frac{1}{s+2} \qquad h(t) = e^{-t}u(t) \Leftrightarrow \frac{1}{s+1}$$

$$y(t) = f(t) * h(t) = e^{-2t}u(t) * e^{-t}u(t) \Leftrightarrow Y(s) = F(s)H(s)$$

$$Y(s) = \frac{1}{(s+2)}\frac{1}{(s+1)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right) \Leftrightarrow y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

Solution of Differential and Integro-Differential Equations - 1

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=\frac{df}{dt}+f \quad \therefore \quad y(0^-)=2\,,\ \ y'(0^-)=1\,,\ \ f(t)=e^{-4t}u(t)$$

$$\frac{dy}{dt}\Leftrightarrow sY-y(0^-)=sY-2 \quad \therefore \quad \frac{d^2y}{dt^2}\Leftrightarrow s^2Y-sy(0^-)-y'(0^-)=s^2Y-2s-1$$

$$F(s)=\frac{1}{s+4} \quad \therefore \quad \frac{df}{dt}\Leftrightarrow sF-f(0^-)=\frac{s}{s+4}-0=\frac{s}{s+4}$$

$$(s^2Y-2s-1)+5(sY-2)+6Y=\frac{s}{s+4}+\frac{1}{s+4}$$

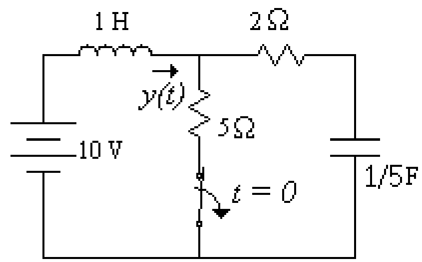
$$(s^2+5s+6)Y-(2s+11)=\frac{s+1}{s+4}\Rightarrow(s^2+5s+6)Y=\underbrace{(2s+11)}_{\text{initial condition terms}}+\underbrace{\frac{s+1}{s+4}}_{\text{input terms}}$$

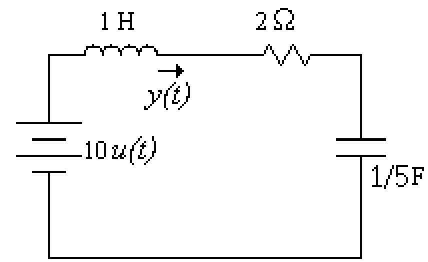
$$Y(s)=\underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{zero-input component}}+\underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{zero-state component}}=\left[\frac{7}{s+2}-\frac{5}{s+3}\right]+\left[\frac{-1/2}{s+2}+\frac{2}{s+3}-\frac{3/2}{s+4}\right]$$

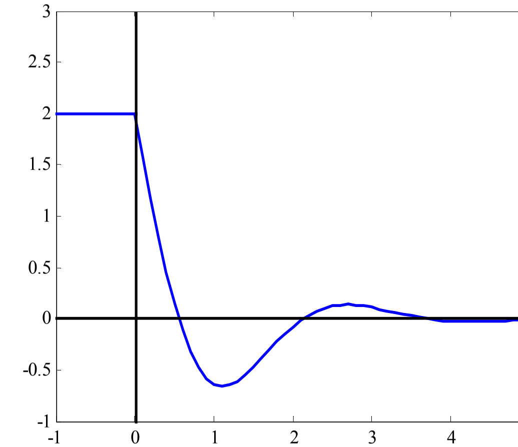
$$y(t)=\underbrace{(7e^{-2t}-5e^{-3t})u(t)}_{\text{zero-input response}}+\underbrace{\left(-\tfrac{1}{2}e^{-2t}+2e^{-3t}-\tfrac{3}{2}e^{-4t}\right)u(t)}_{\text{zero-state response}}$$

$$y(t)=\underbrace{\left(\tfrac{13}{2}e^{-2t}-3e^{-3t}\right)u(t)}_{\text{natural-response}}+\underbrace{\left(-\tfrac{3}{2}e^{-4t}\right)u(t)}_{\text{forced-response}}=\left(\tfrac{13}{2}e^{-2t}-3e^{-3t}-\tfrac{3}{2}e^{-4t}\right)u(t)$$

Solution of Differential and Integro-Differential Equations - 2







$$\frac{dy}{dt} + 2y + y\int_{-\infty}^{t} y(\tau)d\tau = 10u(t) \;\therefore\; \frac{dy}{dt} \Leftrightarrow sY - y(0^-) = sY - 2$$

$$\int_{-\infty}^{t} y(\tau)d\tau \Leftrightarrow \frac{Y}{s} + \frac{\int_{-\infty}^{0^-} y(\tau)d\tau}{s} = \frac{Y}{s} + \frac{Cv_c(0^-)}{s} = \frac{Y}{s} + \frac{\frac{1}{5}(10)}{s} = \frac{Y}{s} + \frac{2}{s}$$

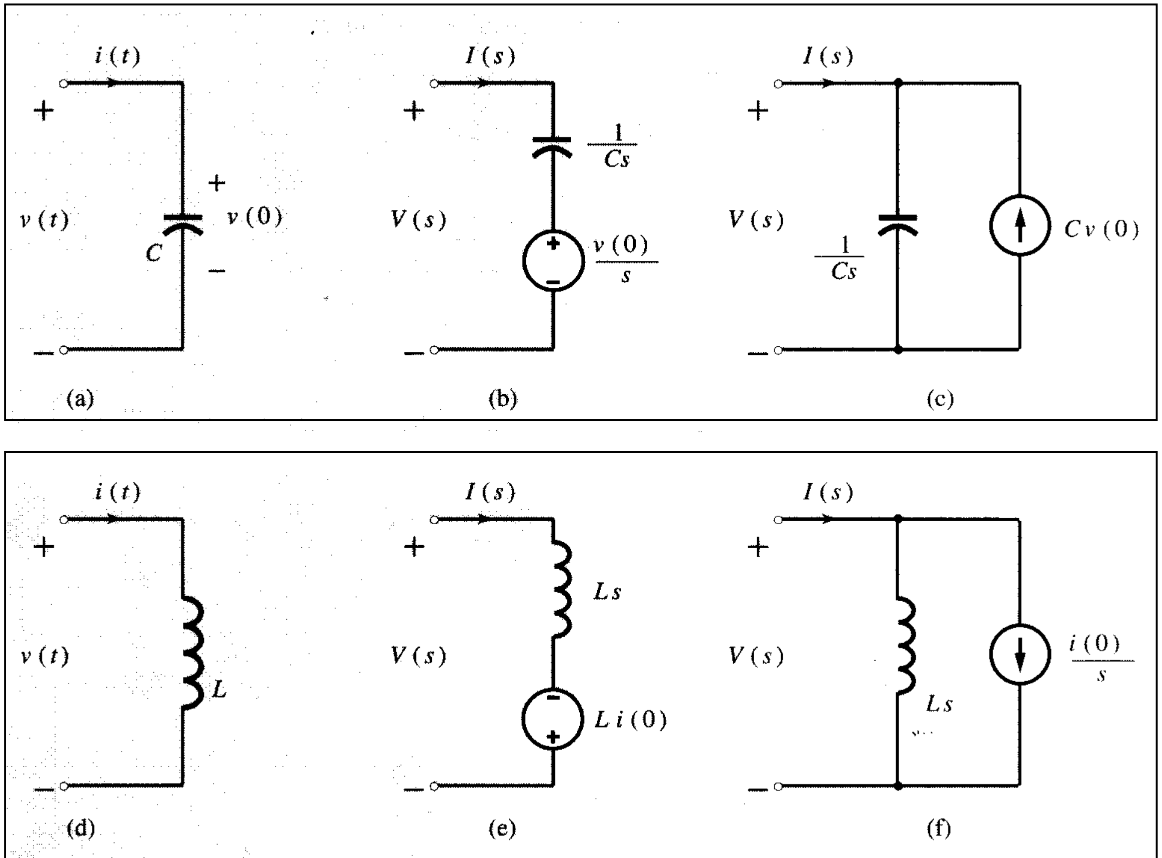
$$sY - 2 + 2Y + \frac{5Y}{s} + \frac{10}{s} = \frac{10}{s} \Rightarrow (s^2 + 2s + 5)Y = \underbrace{(2s - 10)}_{\text{initial condition terms}} + \underbrace{10}_{\text{input terms}}$$

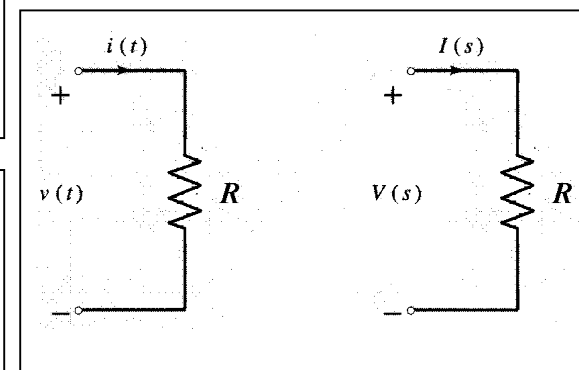
$$Y(s) = \underbrace{\frac{2s}{s^2 + 2s + 5} - \frac{10}{s^2 + 2s + 5}}_{\text{zero-input component}} + \underbrace{\frac{10}{s^2 + 2s + 5}}_{\text{zero-state component}}$$

$$y(t) = \underbrace{\left[\sqrt{5}e^{-t}cos(2t + 26.6^o) - 5e^{-t}sin(2t)\right]u(t)}_{\text{zero-input response}} + \underbrace{\left[5e^{-t}sin(2t)\right]u(t)}_{\text{zero-state response}}$$

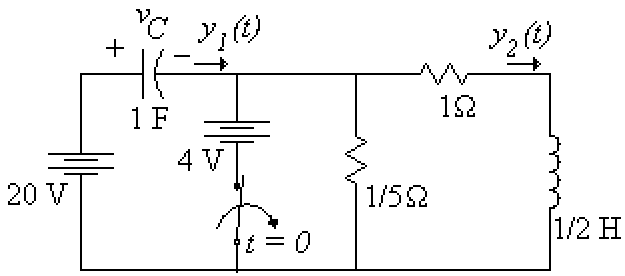
$$y(t) = \underbrace{\left[\sqrt{5}e^{-t}cos(2t + 26.6^o)\right]u(t)}_{\text{natural-response}} + \underbrace{0}_{\text{forced-response}} = \left[\sqrt{5}e^{-t}cos(2t + 26.6^o)\right]u(t)$$

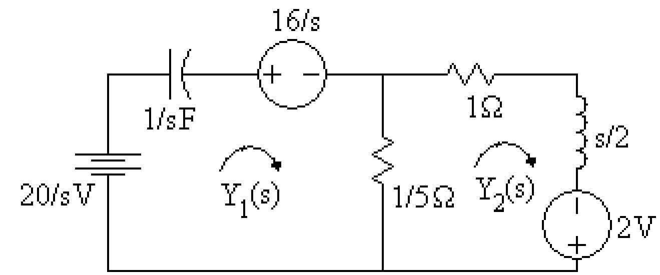
The Analysis of Electrical Networks:
The Transformed Network





The Transformed Network: Example





$$\frac{Y_1(s)}{s} + \frac{1}{5}\left[Y_1(s) - Y_2(s)\right] = \frac{4}{s}$$

$$-\frac{1}{5}Y_1(s) + \frac{6}{5}Y_2(s) + \frac{s}{2}Y_2(s) = 2$$

$$\begin{bmatrix} \frac{1}{s} + \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{6}{5} + \frac{s}{2} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 2 \end{bmatrix}$$

$$Y_1(s) = \frac{24(s+2)}{s^2 + 7s + 12}$$

$$Y_1(s) = \frac{24(s+2)}{(s+3)(s+4)} = \frac{-24}{s+3} + \frac{48}{s+4}$$

$$y_1(t) = (-24e^{-3t} + 48e^{-4t})u(t)$$

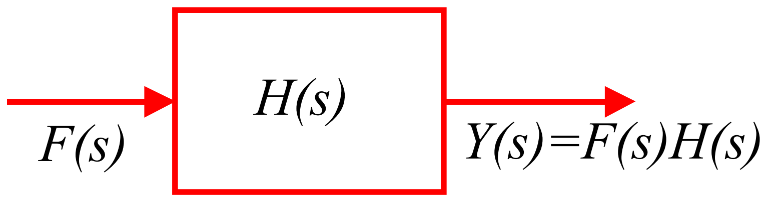
$$Y_2(s) = \frac{4(s+7)}{s^2 + 7s + 12}$$

$$Y_2(s) = \frac{4(s+7)}{(s+3)(s+4)} = \frac{16}{s+3} - \frac{12}{s+4}$$

$$y_2(t) = (16e^{-3t} - 12e^{-4t})u(t)$$

Zero-State Response:

The Transfer Function of an LTIC System



$$H(s) = \frac{P(s)}{Q(s)}$$

The Transfer Function is the ratio of *Y(s)* to *F(s)* when all the initial conditions are zero (when the system is in zero state):

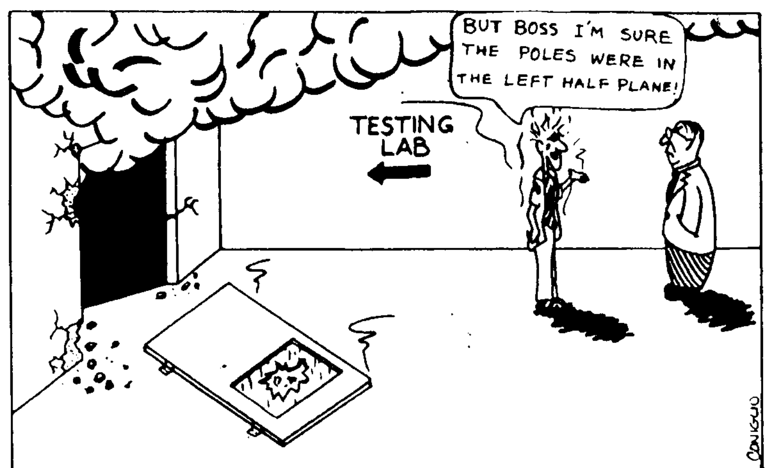
$$H(s) = \frac{Y(s)}{F(s)} = \frac{L[\text{zero} - \text{state response}]}{L[\text{input}]}$$

P(s) and *Q(s)* are polynomial in "*s*". The *Q(s)* is called characteristic equation of the system. The roots of *Q(s)* and *P(s)* are called "poles" and "zeros" respectively of the system. The system's order is given by the number of poles (*n*=order). Therefore, the poles of *H(s)* are the characteristic roots of the system.

Using other approach, *H(s)* can also be defined as the Laplace Transform of the impulse-response when the system is in zero-state.

The System Stability:
The Transfer Function of an LTIC System

- ✓ An LTIC system is asymptotically stable if and only if all the poles of its transfer function $H(s)$ are in the LHP. The poles may be repeated or unrepeated.
- ✓ An LTIC system is unstable if and only if either one or both of the following conditions exist: (i) at least one pole of $H(s)$ is in the RHP; (ii) there are repeated poles of $H(s)$ on the imaginary axis.
- ✓ An LTIC system is marginally stable if only if there are no poles of $H(s)$ in the RHP, and there are some unrepeated poles on the imaginary axis.



Transfer Function – Example 1

- ✓ Show that the transfer function of (a) an ideal delay of T seconds is e^{-sT}; (b) an ideal differentiator is s; (c) an ideal integrator is $1/s$.

(a) $y(t) = f(t-T) \xrightarrow{L} Y(s) = F(s)e^{-sT}$

$$H(s) = \frac{Y(s)}{F(s)} = e^{-sT}$$

(b) $y(t) = \dfrac{df}{dt}$

$Y(s) = sF(s) \quad [f(0^-) = 0 \text{ for a causal signal}]$

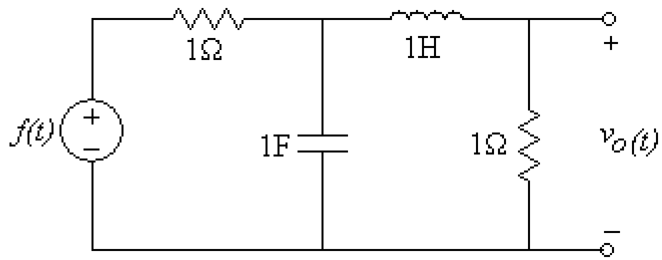
$$H(s) = \frac{Y(s)}{F(s)} = s$$

(c) $y(t) = \int_0^t f(\tau(\tau) \xrightarrow{L} Y(s) = \frac{1}{s}F(s)$

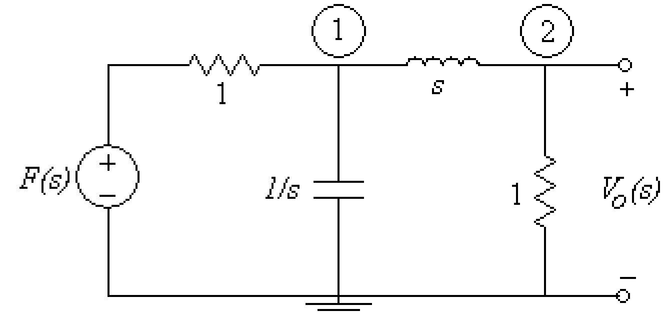
$$H(s) = \frac{1}{s}$$

Transfer Function – Example 2

✓ Find the transfer function relating the output $v_o(t)$ to the input $f(t)$ of the network in fig. below. Find the zero-state response $v_o(t)$ if the input voltage is $f(t)=te^{-t}u(t)$.



$\xrightarrow{L}$



$$(1) \rightarrow \frac{V_1 - F}{1} + \frac{V_1}{1/s} + \frac{V_1 - V_2}{s} = 0$$

$$V_1(1+s+1/s) - V_2(1/s) = F$$

$$(2) \rightarrow \frac{V_2 - V_1}{s} + \frac{V_2}{1} = 0$$

$$-V_1(1/s) + V_2(1/s+1) = 0$$

$$V_2 = V_o = \frac{\begin{vmatrix} 1+s+1/s & F \\ -1/s & 0 \end{vmatrix}}{\begin{vmatrix} 1+s+1/s & -1/s \\ -1/s & 1/s+1 \end{vmatrix}}$$

$$\frac{V_o(s)}{F(s)} = \boxed{H(s) = \frac{1}{s^2+2s+2}}$$

$$f(t) = te^{-t}u(t) \xrightarrow{L} F(s) = \frac{1}{(s+1)^2}$$

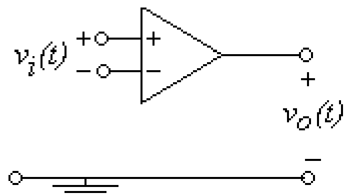
$$V_o(s) = H(s)F(s) = \frac{1}{(s+1)^2}\frac{1}{s^2+2s+2}$$

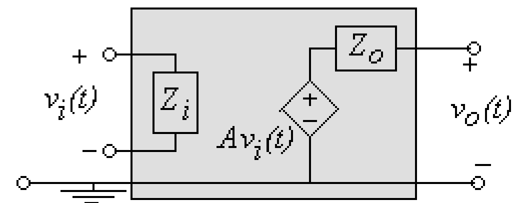
$$V_o(s) = \frac{1}{(s+1)^2} + \frac{0.5e^{j\pi/2}}{s+1-j} + \frac{0.5e^{-j\pi/2}}{s+1+j}$$

$$\boxed{v_o(t) = \left[te^{-t} + e^{-t}\cos\left(t+\pi/2\right)\right] u(t)}$$

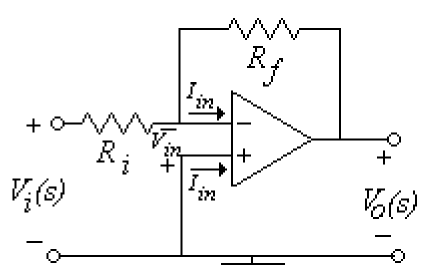
Analysis of Active Circuits

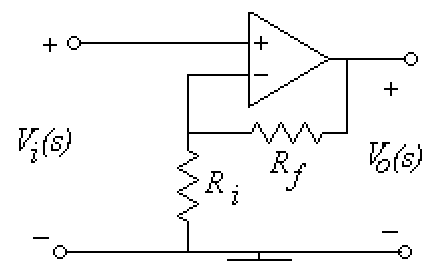
Operational Amplifier and its equivalent circuit.





A typical op amp has a very large gain. The voltage gain *A* is typically 10^5 to 10^6. The output impedance is very high (typically $10^6\Omega$ for BJT to $10^{12}\Omega$ for Bi-FET), and the output impedance is very low (50 to 100Ω). For most applications, we are justified in assuming the gain *A* and the input impedance to be infinite and the output impedance to be zero.



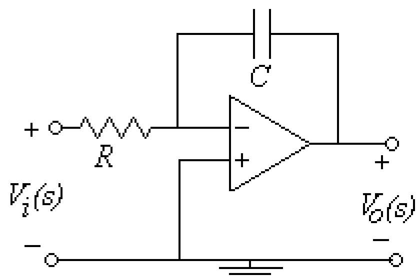


$$V_{in} = 0 \therefore I_{in} = 0 \Rightarrow H(s) = \frac{V_o}{V_i} = -\frac{R_f}{R_i}$$

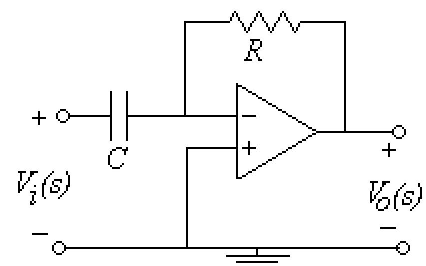
$$H(s) = \frac{V_o}{V_i} = \left(1 + \frac{R_f}{R_i}\right)$$

Analysis of Active Circuits - Exercises

- ✓ Find the transfer function for the circuits shown in figures below.



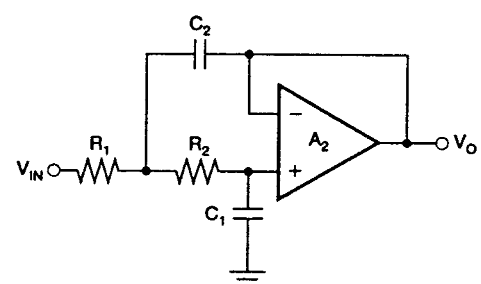
Ans.: (Integrator) $H(s) = \frac{V_o}{V_i} = -\frac{1}{sRC}$



Ans. (Differentiator) $H(s) = \frac{V_o}{V_i} = -sRC$

- ✓ For the circuit shown in figure below calculate C_1 and C_2 knowing that R_1= 3KΩ and R_2= 6KΩ to match with the following transfer function

$$H(s) = \frac{10^6}{s^2 + 50s + 10^6}.$$



Ans.: $C_1 = 5.6\text{nF}$ $\quad C_2 = 10\text{uF}$

Block Diagrams